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Students' Cognition of The Induction Step in Proving Inequality Propositions

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ABSTRACT

The induction step in proof by induction requires some clever tricks in order to get the expected formula for $n = k + 1$, especially in statements with inequalities. The purpose of this study was to determine undergraduate students' cognition regarding the induction step in proving inequality propositions in order to create opportunities to teach the principle of mathematical induction better. A class of 67 students participated in the study on learning proof by induction using the problem-based approach. The Action-Process-Object-Schema theory was used to structure the study and the cyclic activities-class-discussion-exercises instructional approach was used to teach proof by induction. Data for the study was comprised of individual students' written responses to a task of two questions and the transcriptions of the semi-structured interviews. The findings revealed that students at most showed indication of partial understanding of proof by induction. Executing the induction step sits at the heart of proof by induction and necessitates logical reasoning at the object-level conception. Inadvertently, the implication was the most challenging aspect in proof by induction. The majority of students made inroads in setting up the proof properly but could not succeed in proving that $P(k) \Rightarrow P(k+1)$. Some students had challenges of where to begin a proof, so much that they chose to start with direct substitution. In line with that, most students also concluded without deriving the expected formula required to draw a conclusion.
***Keywords**: induction step; proof by induction; APOS-ACE teaching cycles; inequalities.*

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INTRODUCTION

Proofs in mathematics induct to students to logical reasoning and the certainty of theorems. Proofs are systematic arguments that deduce the proposition from other statements that are known to be true. Proving a theory can be a daunting process for students and the logical reasoning and steps that goes with proofs is a skill that has to be learnt. Whilst most mathematics educators consider proofs to be a defining characteristic of mathematics, proofs are known to be difficult for students to master (Inglis, 2011). Stylianides et al. (2007) suggest two reasons for teaching mathematical proofs: they are necessary for a deeper understanding of mathematical concepts and students' skill in doing proofs has potential to foster their mathematics competency.

There are several types of proofs that students learn in undergraduate calculus and these include the principle of mathematical induction, proof by contrapositive, proof by contradiction, checking all cases, proof by counter-example (Dubinsky, 1990) and many others encountered in undergraduate real analysis. The focus of this study is on the induction step of proof by induction. Induction is used to prove that a statement is valid for all cases of $n \in \mathbb{N}$. Proof by induction is used when there are a set of statements $P_1, P_2, P_3, \dots, P_n, \dots$, and there is need to prove that they are all true. It is executed in three steps, namely the basis, the inductive assumption and the induction steps. There are also categories of propositions that require the principle of mathematical induction, but introductory calculus courses focus mainly on three, which "… consists of methods of proof of a mathematical statement in the form of sequence, inequality, and division" (Adinata et al., 2020, p. 1).

In South Africa, deductive proofs commences in grade 11 (fourth year of secondary school) with the Euclidian geometry proofs. Mathematical induction "… is a rigorous form of deductive proof" (Ernest, 1984, p. 181) but is not included in the secondary school mathematics curriculum. Some studies, however, report of teaching proof by induction starting in upper secondary school (Adinata et al., 2020; Palla et al., 2012; National Council of Teachers of Mathematics, 2000) and these studies report that secondary school face various difficulties to understand and apply the principle of mathematical induction in proving propositions. South African students have a delayed start to proof by induction, which only starts in undergraduate education but it is yet to be seen if this helps them to understand induction proof better. There are no studies focusing South African undergraduate students' construction of knowledge for proof by induction.

The nature of proof by induction is unlike what students have seen before in secondary education. Indeed students have proved trigonometric identities and Euclidian geometry. Thus, Dubinsky (1986) posits that proof by induction presents cognitive obstacles and if not addressed, students will continue to struggle with it if instructional methodologies continue to ignore the obstacles. Moreover, Ernest (1984) notes that students have difficulty in producing correct proofs by induction. In most cases students can successfully determine the basis and inductive assumption steps procedurally, without thinking beyond the procedure. The two steps involve determining the $P(1)$ and $P(k)$ respectively. However, the induction step require students to construct the logical reasoning to prove

$P(k+1)$ based on the assumption $P(k)$. The induction step is to show that if $P(k)$ is true, then $P(k+1)$ is true also for an arbitrary $k \in \mathbb{N}$. Moreover, the execution of the induction step is not procedural, but dependent on the proposition to be proved. Hence, it is the induction step that pose great difficulties to students leading to many cognitive challenges. These challenges are more pronounced in proving propositions involving inequalities. Students encounter challenges in coming up with the intricate algebraic manipulations required to arrive at the conclusion that $P(k) \Rightarrow P(k+1)$ for propositions involving inequalities. Some important properties of inequalities have to be applied to prove propositions with inequalities. Moreover, the start of the proof of each problem is not uniform, and the success or failure of any proof depends on it. All these concerns when proving statements with inequalities increases the propensity for students' cognitive errors in the proof by induction.

Inequality propositions require tricks that are really useful pertaining to the principle of mathematical induction. For instance, some tricks for inequalities are as follows: an unequal quantity can be added to both sides of an inequality provided the quantity added to the smaller side is smaller than the quantity added to bigger side; an equal quantity can be added or subtracted to both sides of an inequality without violating the inequality. For instance, given that $y \leq z$ and $m \leq n$, then $y+m \leq z+n$ holds. Similarly, if $y \leq z$ and $m > 0$, then $y \leq z+m$. And, $y+m \leq z+m$ is still true for both $m > 0$ and $m < 0$.

Despite the importance of proof by induction and its application in logic and reasoning in science, engineering, mathematics and technology, research in proof by induction are uncommon. Moreover, among some studies that have been conducted in proof by induction, none specifically focused on the induction step for inequalities propositions. Hence the purpose of this study was to determine undergraduate students' level of understanding of the induction step in proving inequality propositions in order to create opportunities to teach proof by induction in a better way. If students can master the execution of the induction step, then they have understood proof by induction. Basically, the induction step is the pinnacle in proof by induction. Thus, the contribution of this study is to provide insight into how undergraduate students resolve the induction step. The research question for this study was: "What are the cognitive errors displayed by undergraduate students on the induction step when proving inequality propositions?"

To answer to the research questions, the Action-Process-Object-Schema (APOS) theoretical framework and the Activities, Class discussion and Exercises (ACE) teaching cycles were engaged. These are explained in detail in the next section. Many studies have used APOS theory in calculus and other mathematical concepts (Borji et al., 2018; Borji & Voskoglou, 2017) but the ACE teaching cycles have not been widely used. Most APOS studies' focus has been on evaluating students' understanding of a concept only, while excluding how instruction can impact students' understanding of a mathematical concept. The dearth of the APOS-ACE instructional strategy in research was the motivation to explore the learning and teaching of the induction step for inequality propositions.

LITERATURE REVIEW AND THEORY

Whilst there are many studies on proofs and reasoning in mathematics, literature on the proof by induction is scarce and most of it is outdated. One of the earliest studies on proof by induction using APOS theory was by Dubinsky (1986, 1990), where he developed a succinct genetic decomposition for teaching the principle of mathematical induction. Like in most APOS studies, Dubinsky taught proof by induction to a class following the precepts in the genetic decomposition by means of the ACE teaching cycles to a class of 40 students. The method of instruction was effective and led to improvements in students' performance in all the miscellaneous problems that were posed. Hence, given the appropriate teaching methodologies, undergraduate students can successfully overcome cognitive obstacles in learning proof by induction (Dubinsky, 1990). Dubinsky further alludes that once students grasp the idea of proof by induction, they subsequently succeed in proving all types of propositions. In a task with 400 questions given to students to solve, slightly over half of the problems were done correctly. Dubinsky (1990, p. 17) analysed and categorised students' errors in proof by induction as follows: (a) unable to determine the proposition-valued function P and/or incorrect or missing interpretation of the value of P(n); (b) omits consideration of the base case; (c) sets up induction proof properly but does not succeed in proving that P(n) ⇒ P(n+1); (d) miscellaneous error that is serious but does not make the entire solution worthless; (e) problem omitted or nothing much of value in the solution. The categories of errors above are typical of common errors by undergraduate students as they learn different categories of propositions in proof by induction.

In teaching the principle of mathematical induction, instructors often use multiple theorems or propositions. Three most popular propositions used by instructors when introducing proof by induction to secondary and post-secondary students (Hine, 2017; Author, 2021) are general series, divisibility and inequalities. The first two are rather procedural and predictable hence student can do with action-level conception to evaluate the implication $P(k)$ to $P(k+1)$. For instance, Author (2021) reveal that the majority of students correctly used the idea of the proof by induction to prove theorems on divisibilities/multiples. To prove that a statement is divisible by a given number, it suffices to do a direct substitution of $n = k + 1$ in the formula for the induction step. This is not the case with inequalities which require intricate non-standard algebraic manipulations to resolve the induction step. On closer inspection, it is evident that students do not attain object-level conception in the principle of mathematical induction hence they have serious challenges with the induction step with propositions involving inequalities. Hine (2017) explains the three common types of the principle of mathematical induction by giving examples of how to prove each. However, being a conceptual paper, Hine did not elaborated how students would conceptualise the principle of mathematical induction and how they would overcome possible challenges. Furthermore, despite students being able to successfully apply proof by induction to the propositions they are accustomed to, they grapple to make sense of why proof by inductions works. Producing a precise proof does not imply that students know the underlying principles of proof by induction. In other words, students learn how to apply the steps of proof by induction procedurally. However, such learning fail to foster deep understanding of the correctness of the proof whereby students fail to prove propositions of a different and challenging nature, according to Ashkenazi and Itzkovitch (2014).

Some authors suppose that the students' constraints in proof by induction may be due to limitations of teacher knowledge on limits. In that regard, Stylianides et al. (2007) investigated pre-service and secondary school teachers' knowledge of the principle of mathematical induction with a view to determine the knowledge on the principle of mathematical induction that mathematics teachers need in order to teach proof by induction. The findings by Stylianides et al. (2007) indicate that pre-service teachers have difficulties identifying the basis step and the meaning attached to the induction step in resolving $P(k) \Longrightarrow P(k+1)$. The two steps are key to success in proving statements by induction. The inductive hypothesis is obvious, requiring insignificant effort. If teachers encounter difficulties with the induction step themselves then it will be worse with their students.

For proof by induction to be successful, the basis case must be true, in turn enables the assumption $n = k$ to hold. The initial value for the basis case is often $n = 1$ since proof by induction is proven for all natural numbers. However, depending on the proposition given, the basis case may be zero or any other natural number. If the basis case is implicit, verifying the given first element is categorised as an action in the APOS framework (García-Martínez & Parraguez, 2017). In this scenario, then students need the skill to figure out the basis case. The basis case becomes a process mental construction since a student must determine this first. The nuance of the basis case between the conceptions of process and action is brought to the fore and investigated by García-Martínez and Parraguez (2017). García-Martínez and Parraguez reveal the significance of the basis step in proof by induction because failure to construct it means that the principle of mathematical induction lacks substance. Hence, students must not under-rate the basis case and omit it in the proof by induction. Therefore, the basis and induction steps are determinants of the success of the application and effectiveness of proof by induction.

Theoretical framework

To investigate how undergraduate students understand the induction step and their construction of knowledge of proof by induction for inequalities propositions, this study used the tenets of the APOS theory both for both instruction and evaluation of learning. As a constructivist theory, APOS focuses on the hierarchical development individuals' of mathematical knowledge through mental constructions and mechanisms that engender the constructions within social context. The actions, processes and objects constitute the mental structures, and these three are organised into a coherent mental structure called schema (Dubinsky & McDonald, 2001). In this regard, a concept is first conceived at the action level of understanding. Actions are transformations of existing knowledge which are determined externally, for example, explicit step-by-step instructions to perform an operation (Dubinsky & McDonald, 2001). A process is an implicit mental constructions which is a result of repeating an action and reflecting upon it. Student who attain the process conception of a transformation are able to reverse and/or predict the steps to a transformation without actually doing them. When students become aware of a process as a totality and can perform further actions and processes to construct new transformations, the student has encapsulated the process. Finally, a schema is a coherent collection of actions, processes and objects as well as other related schema. The four mental structures are connected coherently to form a framework in students' minds that may be called to solve a problem situation involving the mathematics concept under consideration (Dubinsky & McDonald, 2001).

The application of APOS theory for evaluation of research and instruction is cyclic in nature as shown in Figure 1. The APOS theoretical framework is composed of three components, which are the theoretical analysis, design and implementation of instruction, and collection and analysis of data (Asiala et al., 1996). An APOS inquiry starts with the theoretical analysis, which is a researcher's prediction of the likely mental structures of how students learn a mathematical concept. The researcher implores his or her knowledge of the mathematical concept and of research, and his expertise in the teaching and learning of the concept. The theoretical analysis is also called the genetic decomposition, which is regarded preliminary until it is tested empirically (Arnon et al, 2014) as shown in Figure 1. The focus of this study was on the induction step only, and this calls upon students' object mental constructions. In other words, students encapsulate proof by induction by inferring the induction step $n = k + 1$ from the induction assumption $n = k$. For inequalities, they achieve this by manipulating either side of the inequality to get the formula for $n = k + 1$. Great skill is needed to perform the algebraic manipulations of the inequality, taking into cognisance the basis step. The pre-schemas of inequalities, factorisation, laws of indices and simplifying algebraic expressions play a foundational role to successful execution of the induction step in proof by induction.

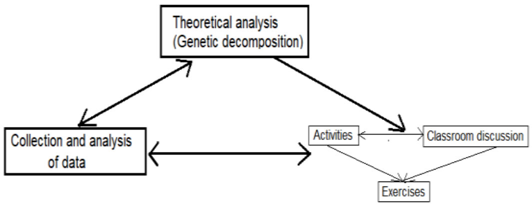

Figure 1. The APOS framework together with the ACE pedagogical approach (modified from Asiala et al., 1996)

The preliminary genetic decomposition guides the design and implementation of instruction, the next step in APOS theory (see Figure 1). The ACE teaching cycles are the instructional approach of the APOS theory. APOS theory is an off-shoot constructivism hence it suggests the ACE teaching cycles as pedagogic approach whereby students construct mathematical knowledge while working and discussing problems. The success or failure of students in resolving the induction step can be identified through the mental constructions that they achieve they solve problems. The instructor attempts to repeat the cycles of activities, class discussions and exercises until students have potentially developed robust mental constructions of the mathematical concept being taught in theory. It is a fact students may not develop expected mental constructions of a concept at the same pace and degree. The methodology section gives further information on the implementation of instruction. The implementation of instruction provides opportunities for gathering and analysing data (Asiala et al., 1996), the last step in the APOS cycle. The results of the preliminary analysis of data may lead to a revision of the initial theoretical analysis and the revised genetic decomposition lays the foundation for the next iteration of APOS theory. The next sections of methodology and findings further give details on the collection of data and analysis of the same.

METHODOLOGY

The descriptive qualitative research methodology was used in this study, which sought to describe existing conditions according to what they were at the time the study was conducted (Yin, 2014). I was the instructor of a second-year calculus course where proof by induction was taught to a class of 67 students. In accordance to the ACE teaching cycle, I issued activities to be done by the students before class on proof by induction. Students worked alone remotely or in groups as blended learning was in effect in 2022 when this study was conducted. Then during class times using Microsoft (MS) Teams, whole class discussions were enacted, which encompassed explanations of the questions done in the activities and sometimes beyond. MS Teams classes lasted for 90 minutes apiece twice in a week. The problems worked on by the students in the activities were discussed during class discussion. As soon as each class ended, homework exercises were administered which consolidated the concepts that were covered in the activities and class discussions. The cycles of activities, class discussions and exercises were repeated until all the concepts under proof by induction were accomplished. The concepts were done in succession starting with the basics of proof by induction, proving general series propositions, multiples propositions and finally inequality propositions. Due to time restraints since the study was part of normal teaching, concepts were not re-taught as such, but the praxis of activities and discussions were. The next step of APOS theory after the implementation of instruction is data collection and analysis. Data-gathering was through a task-sheet consisting of two questions on propositions involving inequalities only. Item 1 was *Prove that for all* $n \geq 3, n^2 > 2n + 3$ *using mathematical induction*, and the second item was *Prove* $4^{n-1} > n^2$ *for* $n \geq 3$ *by mathematical induction.* The items were carefully chosen to enable students to construct expected mental constructions indicated in the genetic decomposition (Ndlovu & Brijlall, 2015).

All the 67 students registered for the calculus course solved the items in the task individually under time control. The written responses were scanned and uploaded to Moodle within the given time period by the individual students. The students' written responses were analysed by paying particular attention to the students' reasoning displayed as they solved the two problems (Radu & Weller, 2011). A further eight students were purposively selected from the initial 67 to take part in the semi-structured interviews. The eight's responses to the task-sheet determined the type and depth of the questions that were to be asked. Interviews sought to get a deeper understanding of the extent to which students attained or failed to attain the object level of mental constructions in doing the induction step for inequality statements.

The interviews were recorded and transcribed by the author for similarities and patterns in students' responses. The first step in the analysis of data involved coding the frequencies for *no* (N), *wrong* (W), *partially-correct* (P) and *correct* (C) responses for each question. This was followed by an in-depth qualitative analysis on the written responses and the interview transcripts to come up with evidence of students' object-level cognition in proving by induction propositions with inequalities. The written responses and transcriptions of the interviews were analysed qualitatively to reveal possible differences in students' performances in specific tasks (Arnon et al., 2014). Failure to solve tasks may indicate that students have not made expected mental constructions while success may mean the mental structures have been made. Students' mental constructions can be deduced from their written and interview responses. For anonymity in this analysis of results, the students were assigned pseudonyms A1, A2 and so on up to A67, but ordering had no significance.

FINDINGS

The frequencies of coded responses are shown in Table 1.

Table 1: The frequencies of students' performance in the task

Response	Question 1	Question 2	Total
No	1	21	22
Wrong	47	30	77
Partial correct	19	16	35
Correct	0	0	0
Total	67	67	134

Question 1 results

All students were eager to solve the problem, except one who did not attempt the question. As can be seen in Table 1, about 70% of the students attempted the question but were incorrect in their proofs. Their greatest weakness was substituting $n = k + 1$ for $n = k$ on both sides of the inequality and simplifying both sides to get varying types of proofs. However, after simplifying both sides, no meaningful conclusions could not be drawn with regard to proof by induction. This was done so by 37 students. Direct substitution is incorrect in proof by induction of inequalities because the statement they got after substitution in the formula happens to be the expected expression for the conclusion to be drawn. Having the expression $2(k + 1) + 3$ on the right-hand side at the beginning is like working backwards from the conclusion. This approach does not also take $P(k) \Longrightarrow P(k + 1)$ into consideration. These students were incognisance of the fact that this implication is the key to successful proof by induction. Figure 2 illustrates this.

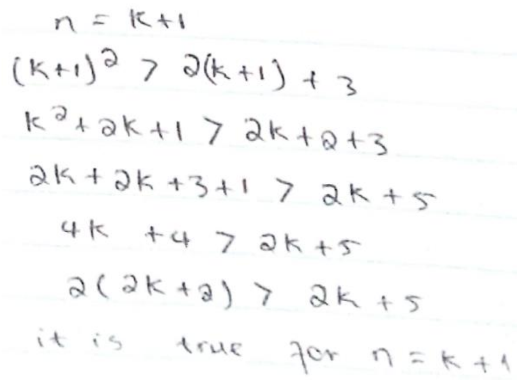

Figure 2. The error of direct substitution in the induction step by A37

Besides those who did a direct substitution, five more students converted the proposition to a quadratic inequality $k^2 > 2k + 3$, which they subsequently solved to get the solution set of k. Three students, A55, A13 and A23 presented written responses which did not consider any of the steps of proof by induction as shown in Figure 3. The question was changed to one of solving the inequality, which they precisely solved. A60 and A4 initially shew some efforts of the basis step and the inductive assumption but then attempted to formulate an inequality which they then made efforts to solve.

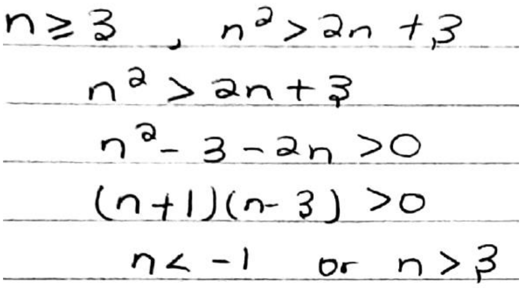

Figure 3. Solving the quadratic inequality instead of proving by induction by A55

Two students, A33 and A31 erroneously used $P(k+2)$ based on inductive assumption of $P(k)$. This was obviously out of sync with domino effect, the underlying principle of proof by induction. The same error of direct substitution was dominant. However, the simplified expressions did not lead to a fruitful conclusion in line with the expectations of proof by induction as shown in Figure 4.

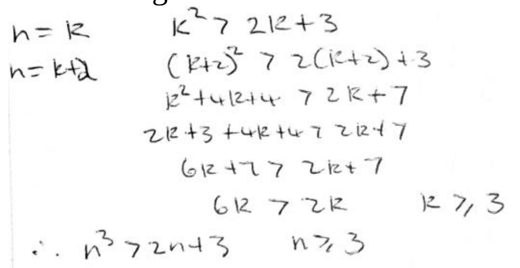

Figure 4. Incorrect substitution of $n = k + 2$ for induction step by A31

Moreover, A51, A56, A57, A64, A65 and A67 substituted both sides of the statement with $n = k + 1$ then applied the inductive assumption to the expansion of $(k+1)^2$. Afterwards they simplified both sides and compared the linear factors to which they concluded that $2x + 10 > 2x + 5$ for $n \geq 3$. Figure 5 illustrates this type of proof which was not even taught in class nor found in texts.

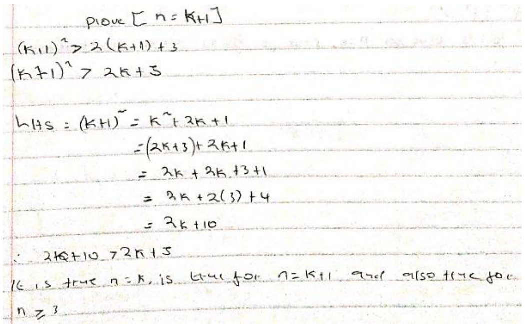

Figure 5. A doubtful induction step by A56.

The proof in Figure 5 is full of doubts because at some point 3 was substituted for k but at one instance only to get $2k + 10$. How was that justified? When a follow-up question was posed to A67, he said he was not sure of what he was doing:

Researcher: Why did you substitute with a three there at second step from the bottom?

A67: The question specified that $n \geq 3$.

Researcher: But then why didn't you substitute in all instances of that k *to get a constant term on both sides?*

A67: I was not sure of what I was doing Sir.

Researcher: Is this how step 3 is done in induction? Did we do so in class?

A67: Uum no. I wasn't sure of the method we used in class either.

In fact if A67 had substituted 3 for k outrightly, he would have obtained $16 > 11$ which is actually the basis step. Hence there is no induction step to talk about.

Again in question 1, about 28% of the students made some progress towards resolving the induction step but encountered some challenges.These students had the correct starting point whereby they expanded $(x+1)^2$ to get $x^2 + 2x + 1$. After this, all of them failed in one or another to perform required algebraic manipulations in order to get the expected formula for $n = k + 1$ on the right-hand side of the inequality. Five students expanded and used the inductive assumption implication to get $4(k+1)$. But instead of de-lineating $2(x+1)+3$, they cancelled by the common factor $(k+1)$ on both sides to solve for k as shown in Figure 6.

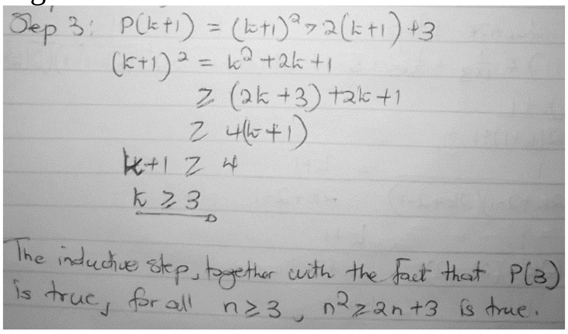

Figure 6. Partially-correct induction step where A58 solved for k

At the end, the conclusion given by A58 did not align to the basic expectations of the principle of mathematical induction. Moreover, some students did not bother to match their conclusion to the original statement as shown Figure 7. They were supposed to get the formula for $n = k + 1$, which is $2(k+1)+3$ to complete the proof by induction. Instead, they got $4(k+1)+3$, which was also incorrect and did not represent the expected statement under the induction step.

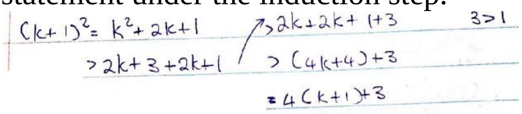

Figure 7. Partially-correct response in the concluding statement by A40.

All students who started with $(x+1)^2$ managed to get $2k+3+2k+1 = 4k+4$ after applying the inductive assumption. However, there was one key reasoning that students were supposed to have done but they all missed it. That step is shown in the complete solution for this problem in Figure 8. None of the students managed to execute the proof shown in Figure 8 or equivalent.

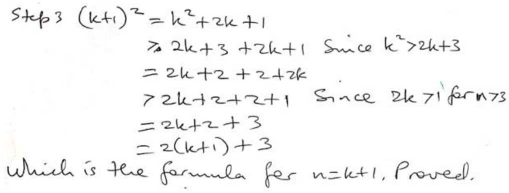

Figure 8. The correct proof for question 1 from the marking guide.

Question 2 results

In the second question, 37 percent of students left the question un-done as shown in Table 1. In most cases, it was because they did not know how to respond or lacked the confidence to attempt the question. When probed in the interviews on why they skipped the induction step, two students said:

A3: I did not know how to start the proof.
A14: It was too difficult for me.

Proof by induction as a deductive proof is systematic, so if students cannot find a suitable starting point for the induction step, then they might not get the courage to do the proof altogether. The frequency of students who attempted question 2 but were incorrect was 45%. Their greatest weakness as in question 1 was to do a direct substitution on both sides of the inequality then do some uncalled for manipulations which do not constitute the principle of mathematical induction. Figure 9 illustrates A20's response by the way of direct substitution. These students could not proceed meaningfully because they started with the expected statement.

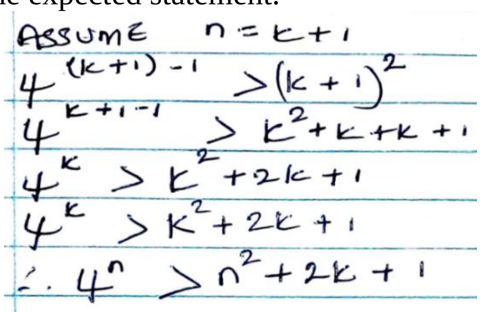

Figure 9. A direct substitution which did

not lead to the expected formula

In the partially-correct responses, 24 percent of the students remembered to multiply by 4 on both sides of the inequality but failed to do the subsequent algebraic simplifications to get the expected formula for $n = k + 1$ in the process. They did not have the knowhow to proceed to the end of the proof after a correct start. Most students correctly deduced the statement $4^k > 4k^2$ but simply ended there as shown in Figure 10. Interestingly, some students also arrived the statement $4^k > 4k^2$ by simplifying $(4^k =)4^{k+1-1}$ to $4^{k-1+1} = 4^{k-1} \times 4 > 4k^2$.

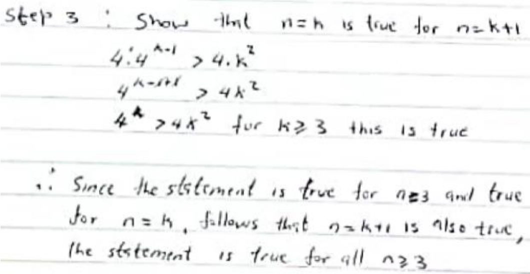

Figure 10. An incomplete induction step by A11

Five more students managed to go past the expression $4^k > 4k^2$. Of these, three made the proof so simple by jumping straight to the concluding statement $4^k > (k+1)^2$ without showing intermediate statements. They did not give the justification of their statements, which casts a doubt if they really knew the logic behind their proofs (illustrated in Figure 11).

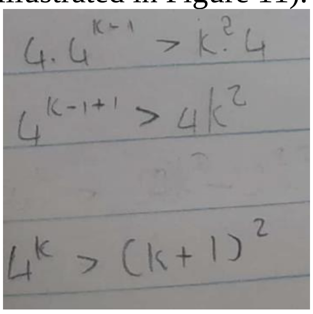

Figure 11. A simplified proof which lacks intermediate statements reasoning by A40

The remaining two students managed to show the correct intermediate statements to get $4^k > (k+1)^2$ but the justification as it relates to inequalities was not given. Figure 12 depicts A56's missing reasoning. The proof in Figure 8 made reference to the inductive hypothesis (written I.H) which was not applicable here. The correct reasoning to justify $k^2 + 2k^2 + k^2 > \; k^2 + 2k + 1$ is the fact that $2k^2 > 2k$ and $k^2 > 1$ for $n \geq 3$.

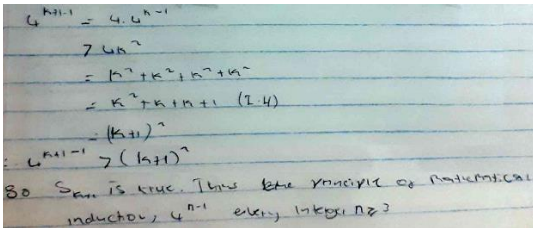

Figure 12. A partially-correct proof by A56 which lacked reasons

Another faulty proof was provided by A5 and A16, whereby they substituted directly $n = k + 1$ and then attempted to perform algebraic manipulations of the right-hand side. But their proofs had three flaws. Firstly, they did not justify why $k^2 + 2k^2 + k^2$ became $k^2 + 2k + 1$. Secondly, the relationship is not always true: $k^2 + 2k^2 + k^2 > \; k^2 + 2k + 1 > (k+1)^2$. The problem lie with the fact that originally $4k^2 = k^2 + 2k^2 + k^2$was greater than $(k+1)^2$. Now there is no guarantee that $k^2 + 2k + 1$ which is less than $4k^2$ is absolutely greater than $(k+1)^2$. It is like saying if $a > b$ and $a > c$ then $b > c$. Finally, the final expression is obviously untrue since a term cannot be greater than itself. Figure 13 depicts these flaws.

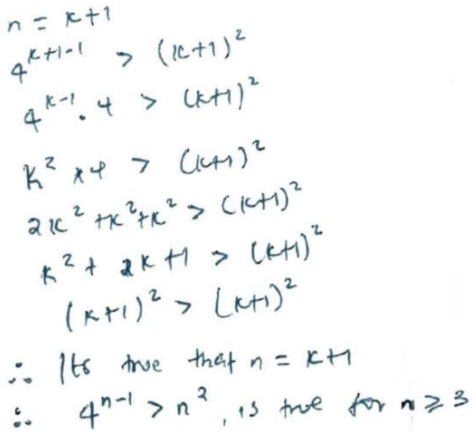

Figure 13. Induction step with flaws by A16

DISCUSSION

In the questions analysed, the total reasoning for the induction step was missing entirely. This indicates that there is no generalisation of the proof of the induction step as in the case of the basis and inductive assumption steps. There is always a unique way of manipulating the algebra to arrive at the expected formula for $n = k + 1$ and the special properties of inequalities comes to the fore. To help students encapsulate the induction step in proof by induction, practice is key. The problem-based approach of the ACE teaching cycles suggested and used in this study aptly provided the much needed practice. The students' performance in the task-sheet and interviews revealed that they did not master the algebraic manipulations to get the $P(k+1)$ from $P(k)$ for inequality propositions.

Proving inequality propositions by induction relies on the transitive property of inequalities, which states that if $a < b$ and $b < c$ then $a < c$. The principle of mathematical induction is used in proving propositions of the form $\forall nP(n)$, where the domain is a set of natural numbers of the form of $\{k, k+1, k+2, \ldots\}$. To prove a proposition by induction requires two steps. The first is verifying the basis step $P(1)$. However, if the domain is bounded by k, the basis step becomes $P(k)$. The basis step is often easy and straightforward, or rather trivial, hence it is not part of this study. The induction step comes next, which starts by assuming the induction hypothesis $P(k)$ to be true. According to the APOS theory, determining the inductive assumption is an action conception, which is trivial. Now, the truth of $P(k)$ is used to show that $P(k+1)$ must also be true for the given domain. To prove an implication, all that is needed is to show is that if the antecedent is true then the condition is true also. This rule of logic represents an interiorisation into the *modus ponens* process conception (Dubinsky, 1986; Author, 2021). According to Dubinsky (1986, p. 308), "This last step is the most important but it seems that rarely do under-graduates even reach the point of being able to explain induction." Students are expected to see the implication in totality, that is, as a cognitive object. The analysis of the induction step is corroborated the findings by Radu and Weller (2011) who said that sometimes the relationship between $P(k)$ and $P(k+1)$ is easily in-discernible to many students. Radu and Weller further hypothesised that the type of students' reasoning when proving may depend on the contextual features of the proposition being proved. This study focused on the induction step of inequality propositions alone as it was regarded as difficult (Dubinsky, 1986) and the findings revealed that students' cognition in them was inadequate. The construction of knowledge for the implication of $P(k)$ to $P(k+1)$ does not occur spontaneously for most students.

One way to stimulate the students to learn mathematical concepts is through teaching which must include plenty of activities. The ACE teaching cycles accord students practice with large of number of examples. The application of the ACE teaching cycles require that the mathematical concept under consideration be divided into smaller sub-topics and each iteration of the cycle corresponds to one of the sub-topics. The activities done before and after class, as well as the whole class discussion form the first cycle of the ACE approach, whose goal is to foster the growth of the appropriate mental structures in the minds of the students (Voskoglou, 2013). Guided by the genetic decomposition, instructors ought to design activities that will help students perform reflective abstractions necessary to construct the necessary mental constructions (Dubinsky, 1990). To construct mental constructions, students must be engaged and cognitively active with content. In this study, it was the case that after students engaged with problems on the concept of proof by induction, they could not succeed applying this method to prove inequality propositions.

For a students to be able to think of an implication as an object, they should be able to interpret the same mathematical entity in at least one way. For instance, students approached the induction step of question 2 of the task-sheet in two ways. Firstly, some multiplied both sides of the inductive assumption by 4 to obtain

$$4^k > 4k^2$$
$$= k^2 + 2k^2 + k^2$$
$$> k^2 + 2k + 1 \ \ since\ 2k^2 > 2\ and\ k^2 > 1\ for\ k \geq 3$$
$$= (k+1)^2$$

Therefore, it true for $n = k + 1$, which completes the proof.

Other students had a different approach to the same proof. They realised that the expected formula for $n = k + 1$ is $4^k > (k+1)^2$. Thus, they started with the left-hand side expression 4^k. Then, $4^k = 4^{k-1+1} = 4^{k-1} \times 4 > k^2 \times 4\ by\ the\ inductive\ assumption$

The two ways of proving becomes similar at the stage where $4^k > 4k^2$, so that the rest of the proof is identical. All students were taught in the same class hence it took ingenuity for the few to come up with at least one approach to the proof. However, the findings revealed that none of the few students who took the two approaches managed to use the properties of inequalities in conjunction with the universe of discourse. When students possess full objection conception of a mathematical concept, they possess skills too to perform further processes and actions to the proofs. Manipulating inequalities in conjunction with the universe of discourse involves anticipating and predicting the conclusion, which makes it fall under the category of process conception. Since all students failed the process conception of proof by induction of inequalities, the object conception was inadequate.

The premise is that if students engage in tasks, they are reasonably likely to develop the mental constructions that leads to learning given mathematical concepts (Voskoglou, 2013). This approach is the APOS-ACE teaching cycles. The performance of students in this study for both the task-sheet and interviews was limited mainly because the induction step is connected to the object-level conception of the principle of mathematical induction. Developing an object conception of mathematics concepts is not an easy feat and in most cases takes it takes a long time (Salgado & Trigueros, 2015; Trigueros & Martinez-Plannel, 2010). Nevertheless, this study bore testimony that instructional approaches using the APOS-ACE teaching cycles were indeed successful in assisting students to gain deeper understanding of mathematical concepts (Weller et al., 2003). In fact, Dubinsky (1990) posited that students will continue to under-perform as long as the teaching methodologies do not address students' cognitive difficulties.

The study recommends more teaching cycles during instruction whenever students do not seem to make expected mental constructions in a given mathematical concept. However, the time to conduct more teaching cycles may not be available, which sees the implementation of APOS theory being time-consuming (Weyer, 2010). Asiala et al. (1996) allude that students experience cognitive growth as they learn a mathematical concept through successive refinements as the instructor repeatedly cycles through the components of the APOS theory of Figure 1. Lastly, students are more likely to conceptualise mathematical concepts if they are introduced through definitions followed by activities and discussions.

CONCLUSION

This study chronicled the students' shortcomings in resolving the induction step of inequalities propositions. Students were required to think at object level to prove the induction step, whereby they conceive it in totality. They were expected to perform actions and processes to this totality by doing relevant tricks and comparisons to obtain the concluding statement. However, students encountered many cognitive errors on the induction step. The problem-based instructional approach was suggested and implemented to enhance students' understanding of proof by induction.

By placing emphasis on problem-solving as part of students' construction of knowledge, the APOS-ACE teaching cycles help students develop higher hierarchical mental constructions of mathematical concepts. Moreover, the computational nature of mathematics makes this highly likely. As pertaining to the preliminary genetic decomposition for this study, it was adequate as a description of how the students learn the induction step in proof by induction. Therefore, no revisions are necessary to the genetic decomposition.

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