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First-Year Undergraduate Students' Ways of Thinking in Combinatorics

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ABSTRACT

The fundamental counting principle and counting problems, jointly called combinatorics in this study, commence in the last year of secondary school and proceed to undergraduate studies and beyond. The South African secondary school situation reflects the same worldwide scenario where counting principles are regarded as difficult and poorly performed in the final national examinations. The purpose of this study was to explore undergraduate students' mental constructions in solving counting problems against the backdrop of Grade 12 fundamental counting principles. The Action-Process-Object-Schema theory was used to describe undergraduate students' thinking ways in combinatorics. A single case of a first-year class of 182 students was considered in this study, whereby they all wrote a task on combinatorics, and seven were further interviewed. The findings revealed that students were skilled at solving problems involving the counting principles, which was mainly a step-by-step application of the formulae. This conception is at the action level according to APOS theory, but the goal of teaching is to guide students to attain the object's mental conception. Object conception allows for the solving of diverse real-world counting problems and promotes mathematical thinking skills.
Keywords: *combinatorics, counting problems, apos theory, fundamental counting principle, permutations, combinations, thinking ways.*

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INTRODUCTION

Solving counting problems requires an individual to determine the number of ways of arranging or selecting sets of elements to satisfy given requirements in a given situation (CadwalladerOlsker et al., 2012; Awuah & Folson, 2017). The counting principle is a technique for determining the total number of outcomes of a given experiment without direct enumeration (Ekol & Mlotshwa 2022). The most basic technique tools to solve counting problems are the multiplication and the addition principles. The multiplication principle is also called the fundamental counting principle (FCP). For example, if there are n_1 possible outcomes for the first event and n_2 possible outcomes for the second event, then the total possible number of outcomes for both events is given by $n_1 \times n_2$. The FCP can be generalised to any number of events k, giving the total number of outcomes for k events as $n_1 \times n_2 \times n_3 \times ... \times n_k$. The FCP and counting problem are termed combinatorics. Combinatorics is a part of discreet mathematics that is concerned with the enumeration of objects (Perrin, 2006; Lamanna, Gea & Batanero, 2022).

The common techniques for solving counting are permutations and combinations, which are introduced to students in the final year of secondary school. Put differently, mathematicians use factorials to find the number of permutations and combinations of given counting problems. Permutations and combinations can be seen as specific applications to solve counting problems. The formulae are given implicitly and explicitly depending on the level of study. The selection of subsets when the order of selection matters is called permutation and combination when order does not. Permutations and combinations are important concepts of combinatorics. In South Africa, students are taught combinatorics when the order of selection matters, and at university they transition to selections when order does not matter. Secondary school students solve counting problems entirely using the FCP and formulae are excluded. Formulae are introduced in undergraduate studies where students can make sense of formulae and reason about counting (Kimani, Gibbs & Anderson 2013). If the formulas are imposed on students, they have no apparent meaning and students struggle to determine which formula to use under what circumstances.

The concept of combinatorics spans the secondary school and university levels of education. Hence the counting principles knowledge is progressive whereby the latter aspects rely on a successful conceptualisation of the former. However, it is not always the fact that students understand the basics. Yee (2023) after many years of teaching combinatorics in high schools noted that the students lacked a basic understanding of the counting principle. Then, according to Lockwood (2012), as students advance to undergraduate studies, they encounter considerable difficulties with counting problems. As combinatorics transition from the last year of high school to university, it becomes increasingly "complex and subtle. It is then considered a difficult subject, particularly because it demands more than rote application of formula, making understanding difficult" (Perrin, 2006, p. 1).

The South African secondary school situation reflects the same worldwide scenario where counting principles are regarded as difficult and poorly performed in the final national examinations (Awuah & Folson, 2017). The FCP is covered in Grade 12, the exit grade of high school education in South Africa. Grade 12 is the last of the three grades that form part of the Further Education and Training phase and all topics offered therein allow for developing mathematical skills and reasoning in preparation for more abstract topics in Higher Education and Training (Department of Basic Education (DBE), 2011). FCP is learned and taught as a precursor to probability in Grade 12 as reflected in one of the goals for the Grade 12 Curriculum and Assessment Policy Statement (CAPS) is to "Apply the fundamental counting principle to solve probability problems" (DBE, 2011, p. 49). Table 1 illustrates the performance of Grade 12 students in the national examinations for the previous five years, followed by the same students' performance in counting problems for the last three years in Table 2.

Table 1. Five-year performance record for Grade 12 mathematics (extracted from the 2020 Diagnostic report on mathematics by DBE (2023)

Year	Percentage of students who attained 50% or more
2018	21.7
2019	20.2
2020	22.3
2021	22.8
2022	22.1

Table 2. Grade 12 students' performance in counting problems exclusively (extracted from the 2020 Diagnostic Report on Mathematics by DBE (2021, 2022, 2023)

Year	Grade 12 students' performance as a percentage
2020	25
2021	33
2022	33

Counting principles and probability concepts are relatively new to South African students as they were not compulsory in Grade 12 before the curriculum prevailed in 2010 (Awuah & Folson, 2017). The FCP was introduced in Grade 12 and progressed into undergraduate education immediately in the first year of study. Combinatorics has wide applications in probability (Lockwood, 2012). The factorial operator is a highly used Taylor series and convergence of infinite series (Yee, 2023). Despite the usefulness of counting principles as a building block to future concepts in calculus and other mathematics concepts, undergraduate students transition from high school with content gaps in FCP. Students' difficulties manifest when they solve counting problems. Students show an understanding of a concept through the problems they solve in tasks and tests. Santagata and Lee (2019) concur by saying "Knowledge is assessed in the context of common problems that arise in the course of teaching mathematics to students" (p. 35). Therefore the purpose of this study was to explore undergraduate students' mental constructions in solving counting problems against the backdrop of Grade 12 counting principles. The research question for this study was, "What are the first-year undergraduate students' ways of thinking in solving counting problems?" Counting problems can be surprisingly difficult to solve, hence more research is needed that explores students' mental reasoning in solving them (Lockwood, 2012). There is a lack of research to inform the learning and teaching of FCP as students progress from Grade 12 to first-year university studies. Tall (2008) believes that there is a transitional phase from introductory to advanced mathematical thinking.

LITERATURE REVIEW

Some efforts ought to be put in place to curb the well-documented poor performance of South African students in mathematics in the final national examinations (Chikiwa, Westaway & Graven, 2019). Atagana et al. (2010) attribute the poor performance to teachers who lack requisite pedagogical content knowledge in specific topics. In that way, students are bound to inherit learning deficits in those concepts that have the potential to negatively impact their future school performance (Taylor and Taylor, 2013; Graven, 2016). Being a relatively new topic in the South African curriculum, FCP may elude some teachers who may not have been trained to teach it. Thus many teachers never bothered teaching FCP before 2014 when it was an optional topic in Grade 12 mathematics. As evidence of limited teacher knowledge in teaching FCP, the DBE annual Diagnostic reports report the same challenges baffling Grade 12 students from 2014 to the present (Ekol & Mlotshwa, 2022). The 2020 Diagnostic reports suggested, "Teach learners the Fundamental Counting Principle in such a way that they will be able to reason answers, instead of trying to remember rules" (DBE, 2021, p. 194). Teachers are expected to offer support to enable students to overcome such identified weaknesses it does seem to be the case. This is consistent with the findings of Makwakwa's (2012) study that posits that teachers encounter challenges when teaching FCP. Hence, the problems students face in solving counting problems are rooted in weak content and pedagogical content knowledge of those who teach it (Awuah & Folson, 2017).

Among students' difficulties in FCP are the nature of the combinatorial operation and elements to be counted, the condition of repetition, and whether to use the addition or multiplication principle. The textbook approach to FCP is to give students formulae for combinatorics and expect them to apply the formula (Yee, 2023). However, this leads to heavy reliance on formulae for computation without thinking beyond the formula (Kazunga & Bansial, 2017). Consequently, reliance on formulae does not help students determine if a given problem requires the formula for permutations, combinations, or FCP. Students may be confused about whether or not the order of elements is needed, which is the sole determining factor to distinguish permutations and combinations. Students may fail to distinguish between permutation and combination problems as a result of failing to establish the connection between the formula for the FCP and the combinatorial operation. On this basis, the Grade 12 students in South Africa learn FCP without using the combinatorics formula (Andrusiak, 2007). Formulas are introduced when they study the same topic at university after students have developed a strong conceptual understanding of combinatorics. However, a study on Spanish high school students by Batanero et al. (1997) revealed that after instruction, students preferred using formulae to solve counting problems. For instance, given a problem involving the number of ways in which eight runners can finish the race as first, second, and third position, Grade 12 students express the solution as $8 \times 7 \times 6$. At university, students use the permutation formula below $8P3 = \frac{8!}{(8-3)!}$.

The study by Andrusiak (2007, p. 19) summarised the sampling procedures that can add complexity to counting problems; 1. Order is important and objects are not replaced; 2. Order is not important and objects are not replaced, and 3. Objects may or may not be distinguishable. Batanero et al. (1997) also emphasize the error of order, that is, not considering the order of objects when it is necessary or distinguishing the order when it is irrelevant. CadwalladerOlsker et al. (2012) viewed error of order as to whether objects needed to be labeled or not. If the chosen objects need to be labeled, this corresponds to permutations, else it is a combination. In my teaching of FCP at university, I have realized that it would minimize students' errors if they regard an ordered arrangement of identical objects as a combination. Identical objects are indistinguishable hence order is irrelevant.

As stated earlier, the FCP is tantamount to the multiplication principle, however, some problems require the use of both the addition and multiplication principles. For example, if a coin is tossed twice, the event "obtaining at least heads" requires the use of both the addition and multiplication principles. In his study, Yee (2023) posited that students lacked the understanding of whether to use the addition or multiplication principles or both.

To evaluate students' thinking combinatorics, the Action-Process-Object-Schema (APOS) theory was used. APOS theory is part of constructivist learning theory that focuses on an individual's construction of mathematical knowledge in a social context. Individuals learn by applying certain mental mechanisms to build specific mental structures of a mathematical concept. The main mechanisms are interiorisation and encapsulation and the related mental structures are actions, processes, objects, and schemas. An individual then uses the mental structures coherently to solve problem situations connected to the corresponding schema (Dubinsky, 1984). The APOS theory is a monitoring framework to determine the progression of mathematics knowledge and the genetic deconstruction of the topic of counting principles forms the basis of the progression one would want to be conceived by students. The APOS levels are the systematic goals on the pathway toward achieving mathematical proficiency in counting principles. The lecturer would keep track of progress through a reflection of attained levels as demonstrated in a student's work in the assessment. To help students overcome difficulties in solving counting problems, teachers ought to fathom students' cognitive understanding of a mathematical concept (Lockwood, 2012). The APOS framework focuses on the mental constructions in the mind of a student when he attempts to learn a mathematical concept (Arnon et al., 2014).

An action is adhering to step-by-step instructions on how to operate (Dubinsky & Mcdonald, 2001). Herein, students compute given counting problems without thinking beyond the direct recall of formulae. A process construction occurs when actions are repeated and students reflect upon them with an internal mental conception.

Students can distinguish between ordered and unordered arrangements and can predict the nature of solutions by direct application of the formulae. When a student conceives a process as a totality and realizes that explicit or mental transformation can act upon the totality, the student has encapsulated the process into an object (Dubinsky, Weller, McDonald & Brown, 2005). Students perform higher-order reasoning to solve counting problems by reflecting on the formula and then applying actions and processes to any counting problems. Finally, an individual's collection of related actions, processes, and objects is called a schema. A schema may contain other related schemas to form a coherent framework in the student's mind (Dubinsky & Mcdonald, 2001). With a coherent framework, a student can decide which mental structure to engage to deal with a counting problem situation within the scope of the FCP schema. A student uses the schema of FCP to delineate non-combinatorial problems and solve them.

METHODOLOGY

This study uses a case study design, which is an in-depth empirical inquiry of an existing phenomenon within its natural setting (Yin, 2014). A single case study of a class of first-year undergraduate students was considered for this study. All the 182 students registered for an introductory calculus course participated in the study. A formal task and semi-structured interviews were used to generate data for this study. After an initial analysis of student's written responses to the task, seven students were selected purposively for interviews based on their written responses. The interviews were used to clarify students' responses to the counting problems in the task (Ndlovu & Brijlall, 2019) and were audio-recorded. Participants were further probed to gain a deeper understanding of their thinking which undergird their written responses (Kazunga & Bansilal, 2017). For the sake of confidentiality, the participants were given labels *X1*, *X2*, *X3*, and so on until *X182*.

The task contained standard questions on FCP as follows: "In how many ways can 3 mathematics books, 4 history books, 3 chemistry books, and 2 biology books be arranged on a shelf so that all books of the same subjects are together?" and "A team of four is chosen at random from five girls and six boys. In how many ways can the team be chosen if there must be more boys than girls." The task was administered after classes of combinatorics were taught traditionally. The written responses and transcriptions of the interviews were analyzed qualitatively to reveal possible differences in students' performances in specific tasks (Arnon et al., 2014). Failure to solve tasks may indicate that students have not made expected mental constructions while success may mean the mental structures have been made. Students' mental constructions can be deduced from their written and interview responses. The correctness of the students' solutions in counting problems was coded using P for Partially correct responses, B for Blank responses, C for correct responses, and N for Incorrect responses. Frequencies for the codes were tabulated. The content analysis was also done to list and collate, where appropriate, the specific points for incorrect, partial, and correct responses of all students (Asiala et al., 1996). The analysis was based on the identification of patterns and themes in the student's written and verbal responses. This information reveals the students' thinking ways and the possible APOS mental constructions they have attained.

FINDINGS

The coding for P, B, C, and N was carried out and shown in Table 1. The table reveals that students performed very well in question 1 but not so in question 2. More students were incorrect or partially correct in question 2 relative to those who were fully correct. The count for no attempts for both questions was small, an indication that few students operated at the pre-action level. The majority of students eagerly attempted the task, save a few in question 2.

Table 3. Frequency of categories of students' responses to the task

Response	Question 1	Question 2	Total
Correct (C)	160	30	190
Partially correct (P)	11	93	104
Incorrect (N)	11	50	61
Blank (B)	0	9	9
Total	182	182	164

The content analysis in the next section is going to reveal the nature of the correct, partially correct, and incorrect responses shown in Table 3.

Question 1 results

This question was a counting principle based on the basics of Grade 12 mathematics, hence 88% of the students were correct in their solutions and thinking processes. They still had the basics of counting principles including the use of the factorial notation from secondary school mathematics. Students recalled that there were four subjects to consider and books under each category were non-identical, hence the solution $4!\,(3! \times 4! \times 3! \times 2!) = 41472$. In this question, students simply had to recall that n indistinguishable elements can be arranged in $n!$ ways and use the multiplication principle to get the result tangibly or mentally. The example of a correct response is shown in Figure 1.

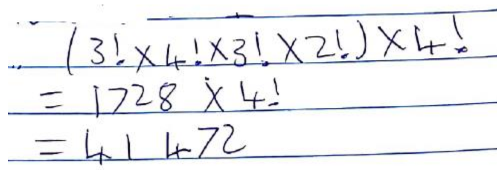

Figure 1. Correct solution showing all steps by X5

A total of 88 percent of the class managed to show reasoning which portrays the attainment of action mental construction. No student operated at the pre-action level as all demonstrated the basic idea of the FCP by attempting to solve using the multiplication principle in one way or another. In the partially correct responses, eight students used the multiplication principle to count ways of arranging books of each type but failed to multiply the result by the number of ways to arrange the four subjects, as shown in Figure 2.

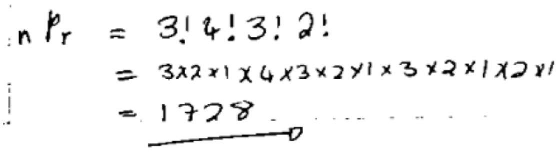

Figure 2. A partially correct solution without the permutation of the four subjects by X28

The other three students made an error of mu I by 3! i of 4! i Itiplication principle. This type of error is a slip, which ais isissporadically and carelessly made by both novices and experts (Ndlovu, Amin & Samuel, 2017). In all, eleven students did not attain the action/process skills in this problem due to flawed approaches to combinatorics. Three students thought the solution was just $12P12$ which they simplified to $12!$, of ways to arrange the 12 books without any restrictions (shown in Figure 3). However, they disregarded the key consideration that all books on the same subjects were to be together.

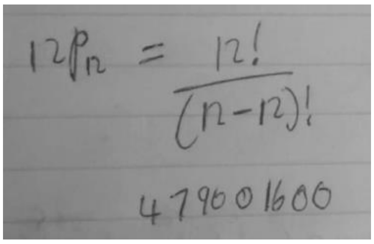

Figure 3. An error in arranging 12 books only by X26

The follow-up interview with X26 confirmed he could relate permutations to factorials:

Researcher : What is the relationship between permutations and factorials very well?

X26 : It is a way of ordering ll n ojec: $Pn = n\ !\ \frac{n!}{(n-r)!}$ if a subset r is selected from the set with $n$elements

Three more students considered all 12 books together but then regarded,,d each of the four types of subject books as identical. In this thinking, they obtained $\frac{12!}{3!\times4!\times3!\times2!}$. Also, the students ccacconted the numberof ways of arranging the four sections for functions 4sections4!, r books osubject A further two students used the addition principle to count ways of arranging each of the books of one subject but oom factorials arranging the four subjects because it could not make sense in addition. Their result was thus $3! + 4! + 3! + 2!$ Finally, one student presented the solution as $3 \times 4 \times 3 \times 2$; he ignored the factorials notation a and4! for arranging the subjects was omitted.

Question 2 Results

The action conception for this question involved using the formula nCr or nPr without thinking much about the type of multiplication principle. In the process cof oinception students distinguish permutations and combinations and use the appropriate formula. The object conception required students to use the multiplication principle ooftwoboysand one girl since the question stated that there must be more boys than girls. After which they perform further transformations mentally or tangibly to get the ssolutiontion00ays. Only 15 percent of the class got the fully correct solution as shown in Figure 4.

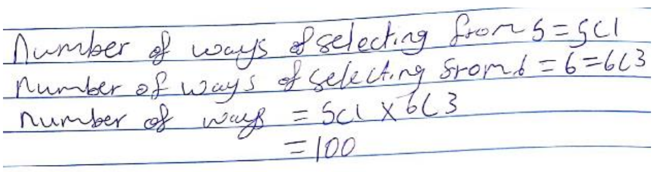

Figure 4. A correct solution which clearly shows the multiplication principle by X5

In the partially correct responses, a total of 80 students interpreted more boys than girls as denoting three boys and one girl or four boys. The use of the addition principle to include only four boys spoiled an otherwise correct solution. Their complete solution was $6C3 \times 5C1 + 6C4 = 20 \times 5 + 15 =$ 115 ways as shown in Figure 5.

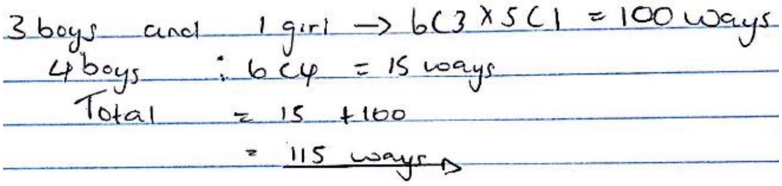

Figure 5. The solution which includes an addition of four boys by X160

Only four boys do not particularly reflect the essence of the question of having fewer girls than boys. The interview extract blow with X80 delves into the matter.

Researcher : Explain your understanding of more boys than girls.
X80 : It means all the possible outcomes must feature.
Researcher : Is that all?
X80 : And four boys.
Researcher : Anymore?
X80 : That's all because I selected a team of four.
Researcher : All right, what can you say about one boy and three girls?
X80 : No no that one is out because boys are fewer than girls.
Researcher : So truly speaking, are there more boys than girls in a team of four boys?
X80 : But we require a team of four Sir.
Researcher : I don't deny that but my point is there must be a team of four while adhering to the demands of the question.
X80 : Ok
Researcher : For example, if I have a team of four boys, do I have a girl on that team?
X80 : No.
Researcher : So what are the possible outcomes again?
X80 : Three boys and one girl.
Researcher : Now what is wrong with your response there?
X80 : I added four boys who should not be there.

Moreover, four students had the same reason as the 80 above but later did not use the addition principle to complete the solution as depicted in Figure 6.

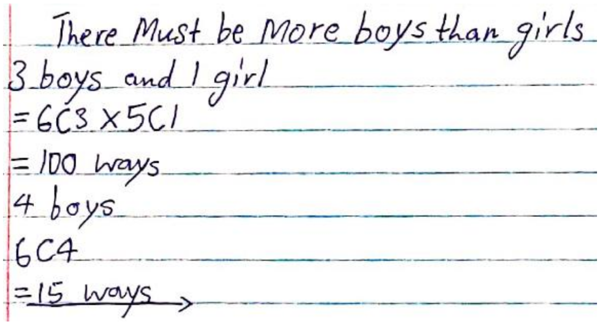

Figure 6. Lack of using addition principle by X129

After determining the required three boys and one girl, X144 used the addition rather than the multiplication principle to obtain $6C3 + 5C1 = 20 + 5 = 25$. This thinking portrays inadequate object conception to distinguish the addition and multiplication principles. Upon further inquiry, X144 realized his error by saying "*I had forgotten that in sets, and denotes multiplication and or denotes addition. I was supposed to write three boys multiplied by one girl*".
The fact that FCP is naturally associated with probability deceived nine students to think of express caps $6C3 \times 5C1$ as a proportion of all selections withoustions e final solution there for eratoras $\frac{6C3\times5C1}{11C4} = \frac{100}{330}$ as illustrated in Figure 7.

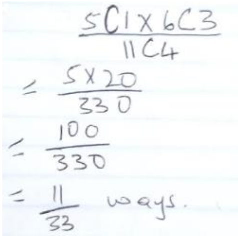

Figure 7. Directly linking FCP to probability by X31.

Twenty-seven percent of the students attempted question 2 but their attempts were futile. Exactly 25 students operated at the action conception whereby they just used the formula to select a team of four from the eleven people. The formula $11C4$ signifies selection without restriction or the sample space, which is at variance with the specifications of the question under consideration. This happens when students use ill-understood formulae, causing them to pick any two numbers to plug into the formula. In this case, they chose $n = 11$ and $r = 4$. Student X166 tried to express her solution as n indicating that she understood order is not important for combinations. However, the order is not important for the four selected members, instead of seven as he wrote. The follow-up interview with X166 revealed that he knew the connection between permutations and combinations:

Researcher : What can you say about selecting $r - like elements from n$.
X166 : It is another way of computing combinations. Permutations of a sample of like elements are the same as combinations.
Researcher : You mean arranging alike elements and selection without order is the same.
X166 : Yes, and even the formula that we use is the same.

In the same reasoning of permutations, X141 mixed permutations and combinations in the same solution by getting $6P3 \times 5Cto$ denote selecting three boys and one girl. With students inclined to use permutations, X63, X56, and X42 understood the question to mean selecting the first, second, and third boy and the first girl as shown in Figure 8.

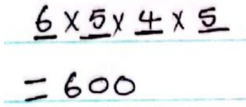

Figure 8. The team of four selected as an ordered arrangement by X56

The interview with X56 revealed his lack of knowledge to distinguish permutations and combinations:

Researcher : What is the difference between permutations and combinations?
X56 : Doc I said I am still halfway in preparation and I need time to complete FCP.
Researcher : But I thought you said you are doing the Grade 12 topics right now.
X56 : Yes counting principles is a Grade 12 topic but it is still too early.
Researcher : Ok. But can you tell me how to tell permutations apart from combinations as you read a given question?
X56 : In combinations, I look for keywords select or choose while in permutations I expect to see the term arrange or something to do with positions.

Also, four students seemingly chose a team of seven as they used the multiplication and addition principles to get $6C5 \times 5C2 + 6C6 \times 5C1$. All four left the expression un-simplified, which could be an indication that they lacked confidence to complete their solutions. It was not clear where they got this information from in a question that specified a team of four but students can conjecture unexpected thinking ways. In addition, they were taught that the lower numbers in the combination formula should add up to the required selection of four whether it is a single or compound event.

In another case of mistaken selection, three students chose one boy and three girls instead of three boys and one girl. After computing the value of $6C1 \times 5C3$ they cresulted in probabilit which was $\frac{6C1 \times 5C3}{11C4} = \frac{60}{330}$. All four produced similar solutions, which raises suspicions of group cheating.

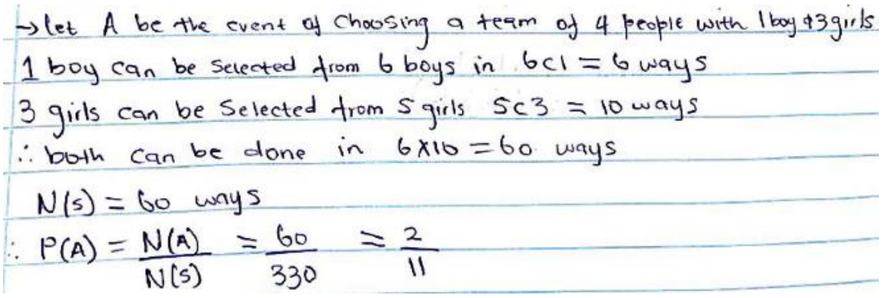

Figure 9. Computing one boy and three girls expressed as a probability

Furthermore, ten students were completely lost in the meaning of what is referred to as selecting a team of more boys than girls. This attempt to use factorials in one or and combinatris One got $\frac{11!}{5!6!}$; another $\frac{11! \times 4!}{5!6!}$; more $\frac{5!+6!}{4!}$ as shown in Figure 10.

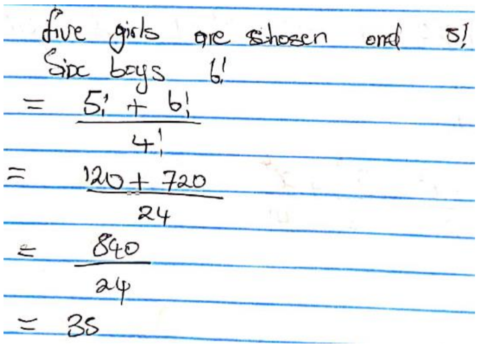

Figure 10. A response with shows a lack of understanding of the question by X125

In the following interview, X125 said the following:

Researcher : Is this how you were taught to solve counting problems?
X125 : I am not good at that Sir.
Researcher : Are you sure?
X125 : I will cover it later. It's too early now. But I will read as I go and become perfect.

X125 indicated that his supposed ignorance of counting problems was just temporary. The interview with another student who was also lost in solving counting problems confirmed X125's experiences.

Researcher : Are you confirming that you don't know the FCP?
X155 : Yes, we are talking facts here Sir.
Researcher : I taught you this topic remember and it's also a Grade 12 topic. How come you find it difficult to solve this counting problem?
X155 : I used to hate chemical equations but now I have to know them and can explain them. When I had to tutor school children here, I had to read and explain it. Maybe the same will happen to the FCP when I have to teach it.
Researcher : As you said you will study and teach it when the time comes, does it mean you allow yourself to fail it now?
X155 : It may happen that way because I am comfortable with other topics in this course.

Students claim to master FCP later when they cannot do the same when the topic is taught in class. Most likely they mean studying for the examination through memorization or as X155 put it, he can get a passing score from topics other than FCP. Finally, nine students were unsure how to answer the question, hence they opted to leave it un-answered.

DISCUSSION

Two counting problems were used in this study so that in solving them, students revealed their thinking ways, as well as manifested the APOS mental construction they had attained in the process of learning FCP. The first question implored students' thinking skills in the counting principle and the second was on combinatorics. The common defining property for both concepts was the use of the factorial notation, a key property to many combinatorial ideas (Lockwood & Erickson, 2016). Most students managed to infuse the factorial notation in the formulae for the counting principle and counting problems, more especially in question 1. The step-by-step usage of formulae is the domain for the action conception, the first level of the APOS theory. Consequently, it was easy for students to attain the action conception because students first understood a concept as an action (Voskoglou, 2015; Mutambara & Tsakeni, 2022). By relying much on externally-driven formula usage, the majority of students could not internalize the mental processes.

A process conception is tantamount to performing the same actions but based on internal stimuli (Arnon et al, 2014). This was evident when students mixed up the multiplication and addition principles, and in some cases failed to alienate all the cases of counting problems and their corresponding applications (Lockwood & De Chenne, 2019). This leads to four types of errors in counting problems, which are the error of repetition ^{n}; error of non-repetition for all elements if they do not get $n!$; t ttheydonot get permutations nPr,and ethe error oofrepetition of a sample without oorderifthey do not get combinations nCr (Batanero & Sanchez, 2013; Batanero et al., 1997; Lockwood, 2012). Repeated elements are technically not part of combinatorics, but are taught in both high schools and universities to clarify students' understanding when learning combinatorics. Unraveling reasoning in combinatorics and subsequent computations of the same whether implicitly or explicitly denote object conception of FCP and counting problems. Question 2 of the task revealed some gaps in the student's reasoning in using the addition and multiplication principles in solving combinatorics. The addition and multiplication principles or both can be used to identify all elements of a compound event. However, students added an extraneous option of selecting four boys in an attempt to exhaust all possibilities. Conversely, students may think they have exhausted all possibilities in a counting problem simply because no other can be found (Kimani, Gibbs & Anderson, 2013). Students' application of the addition and multiplication principles encapsulates their thinking skills in combinatorics. This is in line with literature which reports that developing the object conception in a mathematical concept is the most difficult (Author, 2021; Arnon et al., 2014). The goal of mathematics teaching for students is to encapsulate actions/processes into object conception.

Students oftentimes do not progress to the upper echelons of the APOS theory as a result of the thinking ways they express when they solve problems. For example, students at times try to reduce the learning of counting problems to action reasoning by relying on keywords to distinguish permutations from combinations. Students develop the syndrome "If you see this kind of words, then it is …" (Kimani, Gibbs & Anderson, 2013: **). They stop thinking about mathematics while on the lookout for terms like select/choose or order/arrange to denote combinations and permutations respectively. If approached without thinking, students may be doomed to fail because some of the keywords may be used in a nuanced way in counting problems. In some circumstances, students perform unthoughtful substitutions of numbers they find in a question into the formula for permutations/combinations. This happens when students over-stretch the multiplication principle to solve counting problems. Caddle and Brizuela (2016) posit that students respond to counting problems by identifying two numbers and finding the product.

Combinatorial reasoning is used

to understand how events in a compound operation are formed or selected from a population without manually listing all its elements. Combinatorics are closely related to real-life contexts and use everyday knowledge and language which all individuals undergo at some point in life (Jalan, Nusantara, Subanji & Chandra, 2016). An ordinary person already has an intuition of the FCP in its everyday usage. There are plenty of real-life examples and problems to draw upon which should make sense to all primary, secondary, and tertiary students alike (Rycroft-Smith, Macey, & Rushton, 2020). For small values of n, the permutations can be enumerated directly; but for larger values of, listing all becomes tedious and time-consuming (Yee, 2023). A systematic method for obtaining all the possibilities becomes imperative. Nevertheless, students find combinatorics reasoning difficult (Lockwood & Erickson, 2016; Makinde, 2014; Jalan, Nusantara, Subanji & Chandra, 2016; Lockwood, Swinyard & Caughman, 2015) despite its real-life application and availability of techniques such as tree diagrams which can reinforce this kind of reasoning (Batanero et al., 2016). Students grapple with counting problems, which emphasizes the challenging nature of the topic (Lockwood & Reed, 2020).

Data for this study was based on students' solutions to the task. The effectiveness of learning and teaching lies in the ability of students to solve problems in given mathematics concepts (Makinde, 2014). According to the APOS theory, an individual's accumulation of knowledge is his tendency to respond and solve mathematical problems (Dubinsky & McDonald, 2001). Moreover, as the individual seeks solutions by reflecting on the given problem(s), he or she forms the mental structures used in describing the problem (Syamuri & Santosa, 2021). When solving problems, students undergo specific thinking processes (Frenke & Kazemi, 2001). Two common thinking cognitive processes in a schema are assimilation and accommodation (Simatwa, 2010). Assimilation is when students integrate new experiences into an existing schema in their minds. Accommodation is a process of integrating new information through adjusting existing schema or creating new schema. In learning combinatorics, students had to both add to and modify the existing schema to match the demands of the combinatorics encountered in the first year of undergraduate studies.

CONCLUSION

FCP offers rich opportunities for students to reason and engage in mathematical thinking as they prepare for the diversity of counting problems and in the subsequent topics that follow like probability and series (Lockwood & Erickson, 2016). In the course of teaching, shortcuts or memorizations should be avoided as they act as prescriptions to permit students to avoid exercising thinking skills in mathematics (Kimani, Gibbs & Anderson, 2013). Rather, models that aid students in visualizing answers and constructing combinatorial configurations should be advocated, for example using tree diagrams in dealing with problems that involve compound events (DBE, 2021; Lamanna, Gea & Batanero, 2022) and drawing boxes to represent positions to be occupied by an element in the selection. Students should be equipped to realize that FCP problems are easily solved by the multiplication principle and counting problems by combinatorics formulae.

As in any mathematical concept, students ought to be able to reason about combinatorics and comprehend the formulae they use rather than mechanically apply them. Combinatorics allows for the solving of diverse real-world counting problems and promotes mathematical thinking skills (Rycroft-Smith, Macey & Rushton, 2020; Lockwood, Swinyard, & Caughman, 2015). The thinking ways of undergraduate students in combinatorics were fittingly explained through the lens of the APOS theory. In this study, students were good at applying the multiplication principle and ordering of distinct elements, which is an action conception. However, they had challenges with selecting objects without order, which is an object conception. As a transitional topic between Grade 12 and first-year university, students assimilated and accommodated new ideas in combinatorics. Combinations were new to students which required novel ways to solve them. To lessen students' woes in distinguishing permutations from combinations, Lockwood and De Chenne (2019) advise that permutations are associated with counting sequences whereas combinations count subsets. In some rare cases, permutations may still involve choosing a subset, but the chosen elements require labeling (CadwalladerOlsker et al., 2012). Choosing elements under combinations needs no labeling. The error of ordering is exacerbated by students' failure to use the multiplication and addition principles in a correct way (Holmberg, 2021). FCP and counting principles are taught in both high school and university as a basis for more complex concepts in probability and series (Lamanna, Gea & Batanero, 2022).

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